# A New Procedure for the Solution of Lifting-Surface Problems

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**A new method for determining the steady and unsteady pressure distributions on lifting surfaces is presented. The method employs a "generalized lift-operator" technique which is found to be a more accurate, versatile, and rapid procedure (requiring less computer time) for inverting the downwash integral equation than the presently used "mode-collocation" method. The generalized lift-operator technique allows the chordwise integration to be carried out analytically, thereby avoiding the difficulties in the presently used method. This new approach has been applied to several two-dimensional, unsteady airfoil problems to demonstrate compatibility with known explicit solutions. Numerical solutions for two rectangular foils of aspect ratio 1 and 2 are also presented. The values show good agreement with experiment and fit in with known trends.**

## Introduction

THE solution of the integral equation for lifting surfaces as it arises from the kernel-function approach has attracted many investigators. However, even with the simplification gained by linearization and by assuming ideal fluid conditions, explicit solutions of the unsteady lifting-surface problem has been possible for only a few foil planforms, viz., the wing of infinite aspect ratio, the wing of vanishing aspect ratio, and wings of circular and elliptical planform in incompressible fluid.

Watkins et al.[1,2] have presented a method for approximate solution of the surface integral equation for a finite span wing of general planform in subsonic flow. The basis for this method is the concept that the general character of the lift distribution can be surmised from the few explicit solutions of the lifting-surface problem. The unknown lift distribution is replaced by a sum of modes selected on this basis, each mode weighted by a constant coefficient to be determined in the solution. Watkins selects the Birnbaum modes to represent the chordwise lift distribution, because that series manifests the proper leading-edge singularity and fulfills the Kutta condition along the trailing edge. The known downwash distribution is then expressed as the sum of definite integrals with the unknown coefficients (of spanwise lift) appearing as factors of the integrals. The integrals are evaluated by a numerical scheme, and the unknown lift coefficients are determined by simple collocation at a number of control points.

At present, this is the commonly used approach to solution of the downwash surface integral equation. It is labeled "Mode Approach in Conjunction with the Collocation Method." Many numerical difficulties arise in this method of solution, in particular with the numerical scheme used for the chordwise integration, where extreme accuracy is required with proper accounting for the leading-edge singularity and the oscillatory nature of the chordwise modes. The step behavior of the kernel function near the high-order singularity with finite "Hadamard" contribution must be carefully determined.

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In the course of studies at Davidson Laboratory adapting unsteady lifting-surface theory to marine propellers, a new method has been developed for the solution of the downwash integral equation.[3] By proper expansion of the kernel function and introduction of the so-called generalized lift operator, the chordwise integration is performed analytically, and thus the numerical solution is greatly simplified. The advantages gained by this new method are 1) the chordwise integration is carried out analytically, thus eliminating the problems described previously; 2) the behavior of the series expansion in propeller theory (or integral representation in hydrofoil theory, see Ref. 4) of the kernel is markedly improved by the use of the generalized lift operator which introduces an additional convergence factor; 3) the number of loading modes required to obtain a given accuracy in the solution over the entire chordlength is reduced; 4) as a consequence of these advantages, the computer time required to solve the downwash integral equation is reduced considerably and subsequent use of the solution is facilitated.

In sum, the Davidson Laboratory studies indicate that the generalized lift-operator approach, which is in fact dictated by the nature of the integral equation itself, is a more accurate, versatile, and rapid procedure than the usual numerical approach for evaluating steady and unsteady pressure distributions on lifting surfaces, and the resultant hydrodynamic forces. This technique has been used in Ref. 3 where the lifting surfaces are the blades of a marine propeller operating in nonuniform inflow, and in Ref. 4 for the case of a deeply submerged, flat, rectangular hydrofoil in steady flow.

The purpose of the present paper is to use this generalized lift-operator technique in solving the downwash integral equation for the case of a hydrofoil of finite aspect ratio and general planform operating in a steady or unsteady flowfield. The technique will also be applied to problems of unsteady airfoil theory which have been solved explicitly, such as the cases of an airfoil advancing with constant velocity through a sinusoidal gust and of heaving and pitching foils.

## Linearized Unsteady Lifting-Surface Theory for Finite Foils

The surface integral equation relating the lift and downwash distributions on a finite foil (or control surface) fully immersed in the flow of an incompressible ideal fluid, as derived by means of the acceleration-potential method with the usual linearizing assumptions (small perturbations, thin surfaces), is well documented.[1,2] The equation can be formally written as

$$W(x,y,z;t) = \int_S\int \Delta p(\xi,\eta,\zeta;t) K(x,y,z;\xi,\eta,\zeta;t) dS \quad (1)$$

where

$\left.\begin{matrix} x,y,z \\ \xi,\eta,\zeta \end{matrix}\right\}$ = Cartesian coordinates of control and loading points, respectively
$t$ = time, sec
$S$ = foil surface, ft$^2$
$W$ = downwash velocity distribution normal to foil surface, fps
$\Delta p$ = unknown loading on foil, lb/ft$^2$ (pressure jump across the lifting surface, i.e., $\Delta p = p_+ - p_-$)
$K$ = kernel function representing the velocity induced on the control point $(x,y,z)$ by an oscillatory load of unit amplitude at $(\xi,\eta,\zeta)$ of the foil, ft/lb sec

The foil and coordinate system are shown in Fig. 1. The foil is of general planform but assumed negligible thickness ($z$ and $\zeta \rightarrow 0$). It is seen that on the surface

$$x = \sigma(y) - c(y) \cos\varphi \qquad \xi = \sigma(\eta) - c(\eta) \cos\theta \quad (2)$$

where

$\sigma(y)$, $\sigma(\eta)$ = distance from $y$ axis to midchord line of foil at a spanwise location of control point and loading point, respectively
$c(y)$, $c(\eta)$ = semichord of foil at corresponding spanwise locations
$\varphi$, $\theta$ = angular chordwise positions of control and loading points, respectively

The kernel $K$ is derived as

$$K = -\frac{1}{4\pi\rho U c_0^2}\left[\lim_{(z-\zeta)\to 0}\left(-\frac{\partial^2}{\partial z^2}\right)\int_{-\infty}^{x}\frac{e^{ik(\tau'-x)}}{R}\,d\tau'\right] \quad (3)$$

where

$R$ = $[(\tau' - \xi)^2 + r^2]^{1/2}$
$r^2$ = $(y - \eta)^2 + (z - \zeta)^2$
$k$ = $\omega c_0/U$ reduced frequency
$\omega$ = frequency
$c_0$ = foil semichord on $x$ axis, ft
$U$ = forward velocity, fps, and all linear dimensions within the brace are fractions of $c_0$
$\rho$ = fluid density

The downwash distribution can be the normal components of any imposed velocity. In the linearized theory, the principle of superposition applies and all flow disturbances can be treated separately; their effects are simply added.

The oscillatory velocity distribution normal to the foil may be expressed as

$$W(x,y,0;t) = W(x,y,0)e^{i\omega t}$$

where $\omega$ is angular frequency. It is easily seen that the pressure on the foil must pulsate with the same angular frequency, so that

$$\Delta p(\xi,\eta,0;t) = \Delta p(\xi,\eta,0)e^{i\omega t}$$

With these two substitutions, Eq. (1) becomes

$$W(x,y,0) = \int_S\int \Delta p(\xi,\eta,0)K(x,y,0;\xi,\eta,0;\omega)dS$$

and, finally, by using the trigonometric transformation (2),

$$W(\varphi,y,0) = \int_0^{\pi}\int_{\eta} c_0 L(\theta,\eta,0)K(\varphi,y,0;\theta,\eta,0;\omega)\sin\theta d\theta d\eta \quad (4)$$

with $L(\theta,\eta,0) = c(\eta)\Delta p(\theta\ \eta,0)$ in lb/ft, and $\eta$ a fraction of $c_0$.

Because of the complexity of the kernel function, a direct solution of the integral is impossible and a numerical solution suitable to high-speed digital computers must be sought. The commonly used method, the mode approach in conjunction with the collocation method, is adequately described in Refs. 1 and 2. The present paper uses the generalized lift-operator technique that depends on the proper expansion of the kernel function. As will be seen in the following section, the separable form of the kernel dictates the form of the lift operator.

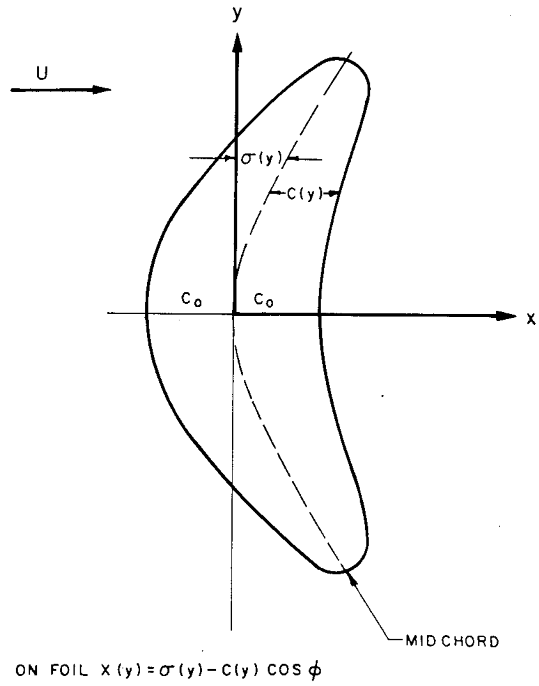



**Fig. 1 Foil and coordinate system.**

## The Separable Form of the Kernel and the Generalized Lift Operator

In Eq. (3), the reciprocal of the Descartes distance $R$ can be expressed as

$$\frac{1}{R} = \frac{1}{\pi}\int_{-\infty}^{\infty} K_0(|\mu|r)e^{i\mu(\tau'-\xi)}\,d\mu \quad (5)$$

where $K_0(x)$ is the modified Bessel function of order zero. Substitution of Eq. (5) in Eq. (3) yields

$$K = -\frac{1}{4\pi^2\rho U c_0^2}\left[\lim_{z-\zeta\to 0}\left(-\frac{\partial^2}{\partial z^2}\right)\times \int_{-\infty}^{x} e^{ik(\tau'-x)}\,d\tau' \int_{-\infty}^{\infty} K_0(|\mu|r)e^{i\mu(\tau'-\xi)}\,d\mu\right] \quad (6)$$

The $\tau'$ integration involves

$$\int_{-\infty}^{x} e^{i(k+\mu)\tau'}\,d\tau' = \pi\delta(k+\mu) - \frac{ie^{i(k+\mu)x}}{k+\mu}$$

where $\delta(\ )$ is the Dirac delta function. The $\mu$ integral is then

$$I_\mu = \int_{-\infty}^{\infty} K_0(|\mu|r)e^{-i\xi\mu}\left[\pi\delta(k+\mu) - \frac{ie^{i(k+\mu)x}}{k+\mu}\right]d\mu = \pi K_0(kr)e^{ik\xi} - ie^{ikx}\int_{-\infty}^{\infty}\frac{K_0(|\mu|r)e^{i\mu(x-\xi)}}{k+\mu}\,d\mu \quad (7)$$

which has an integrable Cauchy-type singularity. After the $z$ derivatives and the limit are taken, with

$$\partial^2 I_\mu/\partial z^2 = (\partial^2 r/\partial z^2)\partial I_\mu/\partial r + (\partial r/\partial z)^2 \partial^2 I_\mu/\partial r^2$$

and

$$\lim_{(z-\zeta)\to 0} (-\partial^2 I_\mu/\partial z^2) = -(1/r)(\partial I_\mu/\partial r)$$

the kernel becomes

$$K = -\frac{1}{4\pi\rho U c_0^2 y_0^2}\left[ k y_0 K_1(k y_0) e^{-ik(x-\xi)} - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{|\mu| y_0 K_1(|\mu| y_0) e^{i\mu(x-\xi)}\, d\mu}{k+\mu}\right] \quad (8)$$

where $y_0 = |y - \eta|$ and is a fraction of $c_0$.

When $y_0 = 0$, the terms within the brackets of Eq. (8) reduce to

$$\left[ e^{-ik(x-\xi)} - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{e^{i\mu(x-\xi)}}{k+\mu}\, d\mu \right] = e^{-ik(x-\xi)}\, [1 + \mathrm{sgn}(x-\xi)]$$

There is, therefore, a high-order singularity in the kernel as $y \to \eta$ and $x - \xi > 0$. In addition, the step behavior of the kernel function as $x - \xi \to 0 \pm$ is obvious. In the steady-state case, the kernel reduces to the well-known results

$$K = -\frac{1}{4\pi\rho U c_0^2 y_0^2}\begin{cases} 2 & \text{when } x - \xi > 0 \\ 0 & \text{when } x - \xi < 0 \end{cases}$$

and, in the unsteady case, when $x - \xi \to 0$,

$$Re\, K = \begin{cases} -2/4\pi\rho U c_0^2 y_0^2 & \text{for } x - \xi \to 0+ \\ 0 & \text{for } x - \xi \to 0- \end{cases}$$

$$Im\, K = 0 \qquad \text{for } x - \xi \to 0\pm$$

It can also be shown that Eq. (8) reduces to the known form of kernel function as expressed in terms of modified Bessel and Struve functions.

With the trigonometric transformation (2), the kernel function becomes

$$K = -\frac{1}{4\pi\rho U c_0^2 y_0^2}\left[ k y_0 K_1(k y_0) e^{-ik(\sigma_i - \sigma_j)} e^{ikc_i \cos\varphi} e^{-ikc_j \cos\theta} - \frac{i}{\pi}\int_{-\infty}^{\infty} \frac{|\mu| y_0 K_1(|\mu| y_0) e^{i\mu(\sigma_i - \sigma_j)} e^{-i\mu c_i \cos\varphi} e^{i\mu c_j \cos\theta}}{k+\mu}\, d\mu \right] \quad (9)$$

where

$$\sigma_i = \sigma(y)/c_0 \qquad \sigma_j = \sigma(\eta)/c_0$$

$$c_i = c(y)/c_0 \qquad c_j = c(\eta)/c_0$$

It should be noted that, in Eqs. (8) and (9), the $x$ or $\varphi$ chordwise dependence of the kernel function (on the control point) and the $\xi$ or $\theta$ dependence (on the loading point) occur in exponential form $e^{\pm i\nu \cos\varphi}$ and $e^{\pm i\nu \cos\theta}$, and thus are separated from each other. The kernel is now in separable "degenerate" form, which not only facilitates the chordwise loading ($\theta$) integration, as will be seen later, but also permits use of the lift-operator technique.

The exponential, $\exp(\pm i\nu \cos\varphi)$, can be expanded in terms of either of the following orthogonal and complete sets of functions, $\Phi(\bar{m})$,

$$1, \cos\varphi, \cos 2\varphi, \ldots, \cos(\bar{m}-1)\varphi, \ldots \qquad 0 \le \varphi \le \pi \quad (a)$$

or

$$(1-\cos\varphi), (1 + 2\cos\varphi), \cos 2\varphi, \ldots, \cos(\bar{m}-1)\varphi, \ldots \qquad 0 \le \varphi \le \pi \quad (b)$$

in the form

$$e^{\pm i\nu\cos\varphi} \equiv J_0(\nu) + 2\sum_{\lambda=1}^{\infty} (-1)^\lambda J_{2\lambda}(\nu)\cos 2\lambda\varphi \mp 2i \sum_{\lambda=1}^{\infty} (-1)^\lambda J_{2\lambda-1}(\nu)\cos(2\lambda-1)\varphi$$

where $J_n(\nu)$ are Bessel functions of the first kind.

The orthogonality property of $\Phi(\bar{m})$ dictates operation on both sides of integral equation (4) by either of the following operators:

$$\frac{1}{\pi}\int_0^\pi \cos\bar{m}\varphi\{\quad\}d\varphi \qquad \bar{m} = 0,1,2,\ldots \quad (c)$$

or

$$\frac{1}{\pi}\int_0^\pi (1-\cos\varphi)\{\quad\}d\varphi \qquad \bar{m} = 1$$

$$\frac{1}{\pi}\int_0^\pi (1+2\cos\varphi)\{\quad\}d\varphi \qquad \bar{m} = 2 \quad (d)$$

$$\frac{1}{\pi}\int_0^\pi \cos(\bar{m}-1)\varphi\{\quad\}d\varphi \qquad \bar{m} > 2$$

The first operator of (d) is the known Glauert lift operator introduced in steady airfoil theory. For this reason, the second set of (d) has been chosen in the present study, as an extension of the concept of lift operator, and, because of its general form, it has been named generalized lift operator.

Note that the left-hand side of the integral equation (4) can be expressed in a Fourier series expansion with complex coefficients $W(y)$ and angular dependence on $\varphi$. After application of the generalized lift operator $\Phi(\bar{m})$ of (d), Eq. (4) becomes

$$\sum_{\bar{m}=1} \bar{W}^{\bar{m}}(y) = \sum_{\bar{m}=1}\int_0^\pi c_0 \int_\eta L(\theta,\eta)\bar{K}^{(\bar{m})}\sin\theta\, d\eta\, d\theta \quad (10)$$

where

$$\bar{W}^{(\bar{m})}(y) = \frac{1}{\pi}\int_0^\pi \Phi(\bar{m}) W(\varphi,y,0)d\varphi$$

$$\bar{K}^{(\bar{m})} = \frac{1}{\pi}\int_0^\pi \Phi(\bar{m}) K(\varphi,y,0;\, \theta,\, \eta,0;\, \omega)d\varphi \quad (11)$$

and the superscript $\bar{m}$ refers to the order of the lift operator.

## The Pressure Loading Functions

In the numerical solution suggested by Watkins et al.[1,2] for the integral equation, the unknown loading distribution is expressed as a linear combination of preselected functions. In the chordwise direction, the series of functions is taken to be the Birnbaum distribution of two-dimensional theory and in the spanwise direction the series from lifting-line theory.

The Birnbaum distribution is also chosen here for the chordwise distribution because it reproduces the proper leading-edge singularity (square-root singularity) and fulfills the Kutta condition along the trailing edge. Landahl[5] has confirmed that this distribution has the correct edge behavior for a wing with continuous downwash distribution over the entire surface. However, in this paper the unknown spanwise loading components are left to be determined in the solution of the integral equation.

Thus, on introducing the Birnbaum modes,

$$L(\theta,\eta,0) = \frac{1}{\pi}\left[ L^{(1)}(\eta)\Theta(1) + \sum_{\bar{n}=2}^{\infty} L^{(\bar{n})}(\eta)\Theta(\bar{n})\right] \quad (12)$$

where

$L^{(\bar n)}(\eta)$ = spanwise loading components
$\Theta(1)$ = $\cot(\theta/2)$
$\Theta(\bar n)$ = $\sin(\bar n - 1)\theta,\ \bar n > 1$

the chordwise integration can be performed analytically since, as was shown before, the $\theta$ dependence of the kernel is expressed in exponential form as $e^{\pm i\nu \cos\theta}$ [see Eq. (9)]. After the $\theta$ integration, Eq. (10) is reduced to

$$\sum_{\bar m=1} \bar{W}^{(\bar m)}(y) = \sum_{\bar m=1}\sum_{\bar n=1} c_0 \int_\eta L^{(\bar n)}(\eta)\bar{K}^{(\bar m,\bar n)}d\eta \quad (13)$$

where the modified kernel is

$$\bar{K}^{(\;,n)} = \frac{1}{\pi^2}\int_0^\pi\int_0^\pi \Theta(\bar n)\Phi(\bar m)\ K(\varphi,y,0;\theta,\eta,0;\omega)\times$$

$$\sin\theta d\theta d\varphi = -\frac{1}{4\pi\rho U c_0^2 y_0^2}\left[k y_0 K_1(k y_0)e^{-ik(\sigma_i-\sigma_j)}\times\right.$$

$$I^{(m)}(kc_i)\Lambda^{(n)}(kc_j) - \frac{i}{\pi}\times$$

$$\left.\int_{-\infty}^{\infty}\frac{|\mu|y_0K_1(|\mu|y_0)e^{i\mu(\sigma_i-\sigma_j)}\,I^{(\bar m)}(-\mu c_i)\Lambda^{(\bar n)}(-\mu c_j)d\mu}{k+\mu}\right] \quad (14)$$

and

$$I^{(\bar m)}(\nu) = \frac{1}{\pi}\int_0^\pi \Phi(\bar m)e^{i\nu\cos\varphi}\,d\varphi$$

$$\Lambda^{(n)}(\nu) = \frac{1}{\pi}\int_0^\pi \Theta(\bar n)e^{-i\nu\cos\theta}\sin\theta d\theta$$

The latter are defined in Appendix A. For $k = 0$ (the steady-state condition),

$$\bar{K}^{(\bar m,\bar n)}(k=0) = -\frac{1}{4\pi\rho U c_0^2 y_0^2}\left[I^{(\bar m)}(0)\Lambda^{(\bar n)}(0) - \right.$$

$$\left.\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{|\mu|y_0}{\mu}K_1(|\mu|y_0)I^{(m)}(-\mu c_i)\Lambda^{(\bar n)}(-\mu c_j)d\mu\right] \quad (15)$$

The singular $\mu$ integral exists in the sense of a Cauchy principal value. Therefore, its finite contribution can be easily determined. The integration is performed by any numerical method such as Simpson's.

Equation (13), to which the original surface integral equation has been reduced, is a set of line integral equations. The number $\bar m$ of the integral equations must be equal to the number $\bar n$ of unknown chordwise modes. The solution of these $\bar m = \bar n$ line integral equations is obtained by the collocation method. If the wing is divided into $J$ strips along the span and the spanwise loading is considered constant over each small strip, then the set of $\bar m$ line integral equations is converted into a system of $\bar m$ simultaneous algebraic equations

$$\bar{W}^{(\bar m)}(y_i) = \sum_{\bar n=1}^{\bar n\,\max}\sum_{j=1}^{J} L^{(\bar n)}(\eta_j)\bar{k}_{ji}^{(\bar m,\bar n)}(y_i,\eta_j) \quad (16)$$

where

$$\bar m = 1,2,3,\ldots,\bar n \qquad j = 1,2,\ldots,J$$

$$i = 1,2,\ldots,J$$

$$\bar{k}_{ji}^{(\bar m,\bar n)}(y_i,\eta_j) = \int_{\eta_j-\beta}^{\eta_j+\beta}\bar{K}^{(\bar m,\bar n)}(y_i,\eta)d\eta$$

and $2\beta$ = length of each spanwise strip.

Through the solution of Eqs. (16), the spanwise loading components $L^{(\bar n)}(\eta_j)$ are determined and the resultant spanwise loading distribution follows from Eq. (12):

$$L(y_i) = \int_0^\pi L(\theta,\eta,0)\sin\theta d\theta = L^{(1)}(y_i) + \frac{1}{2}L^{(2)}(y_i) \quad (17)$$

The integration of $\bar{K}^{(\bar m,\bar n)}$ over all elements of the foil span except that which includes $y = \eta$ is performed by any convenient numerical method. In the region where $y_0 = |y - \eta| \to 0$, the integral has a high-order singularity. Its finite contribution is obtained by using a polynomial approximation of the modified Bessel function valid in a narrow range about $y_0 = 0$ and a three-point Gaussian quadrature for the immediately adjacent ranges.

## Convergence of the Assumed Loading Distribution

The evaluation in Ref. 3 of steady and time-dependent loading distributions on the blades of marine propellers is also based on the approximation of chordwise loading by the Birnbaum distribution of the form

$$L^{(0)}\cot\frac{\theta}{2} + \sum_{\bar n=1}^{\infty} L^{(\bar n)}\sin\bar n\theta$$

The calculations there showed that with increasing number of modes the coefficients $L^{(\bar n)}$ remain of the same order of magnitude (and also tend to a constant value $c$). Since the high-frequency terms do not contribute to the spanwise (integrated) loading distribution,

$$L(y) = L^{(0)}(y) + \tfrac{1}{2}L^{(1)}(y)$$

this distribution converged after a few (three) terms of the Birnbaum distribution were taken. However, the chordwise loading distribution itself showed no sign of convergence even after ten terms of the series were taken.

The cause of the slow convergence of the chordwise distribution is presumed to be the type of assumed chordwise modes. The terms of the Birnbaum series are not linearly independent, as claimed, because the presence of the $\cot(\theta/2)$ term implies that the distribution to be approximated does not belong to $L_2$ (square integrable) on $(0,\pi)$. There exists a sine series which is summable in the Cesaro sense[6] (see Ref. 3) to a $\cot(\theta/2)$ term, viz.,

$$\sum_{\bar n=1}^{\infty}\sin\bar n\theta = \frac{1}{2}\cot\frac{\theta}{2} \quad (18)$$

Therefore, in some sense $\cot(\theta/2)$ and $\sin\bar n\theta$ are linearly dependent, and it is not unreasonable to expect part of the $\cot(\theta/2)$ contribution to be incorporated in the sine series.

It appears from the calculations of Ref. 3 that the coefficients of the sine series are of the form

$$L^{(\bar n)} = a_{\bar n} + c \qquad \text{where} \qquad a_{\bar n} \to 0 \qquad \text{as} \qquad \bar n \to \infty$$

The assumed chordwise distribution can then be written approximately as

$$L^{(0)}\cot\frac{\theta}{2} + \sum_{\bar n=1}^{\infty}a_{\bar n}\sin\bar n\theta + c\sum_{\bar n=1}^{\infty}\sin\bar n\theta =$$

$$\left[L^{(0)} + \frac{c}{2}\right]\cot\frac{\theta}{2} + \sum_{\bar n=1}^{M}[L^{(\bar n)} - c]\sin\bar n\theta \quad (19)$$

since

$$\lim_{\bar n\to\infty}[L^{(\bar n)} - c] = \lim_{\bar n\to\infty}a_{\bar n} = 0$$

This finite sum will converge rapidly. The drawback to using Eq. (19), as it is, is that the constant $c$ must be determined, reasonably closely, by a series of calculations.

Since publication of Ref. 3, a procedure devised at Davidson Laboratory and based on the concept of expression (19) has been built into the solution of the integral equation. This method has secured rapid convergence of the chordwise distribution with the use of a few (five) chordwise modes. If the assumed distribution is rewritten as

$$\left[L^{(0)} + \frac{c}{2}\right]\cot\frac{\theta}{2} + \sum_{\bar{n}=1}^{M}[L^{(\bar{n})} - c]\sin\bar{n}\theta - c\left(\sum_{\bar{n}=1}^{\infty}\sin\bar{n}\theta - \sum_{\bar{n}=1}^{M}\sin\bar{n}\theta\right) \quad \text{(e)}$$

then, after the chordwise $\theta$ and $\varphi$ integrations and the $\eta$-integration of the kernel function over each strip, the right-hand side of Eq. (16) is of the form, for each strip and $\bar{m}$,

$$l^{(0)}\bar{k}^{(\bar{m},0)} + l^{(1)}\bar{k}^{(\bar{m},1)} + \ldots + l^{(n)}\bar{k}^{(\bar{m},n)} - c\left[\frac{1}{2}\bar{k}^{(\bar{m},0)} - \sum_{\bar{n}=1}^{M}\bar{k}^{(m,\bar{n})}\right] \quad \text{(f)}$$

Here

$$l^{(0)} = L^{(0)} + (c/2) \qquad l^{(n)} = L^{(n)} - c$$

are directly determined by the solution and the last loading component $c$ evaluated in the solution is omitted in the chordwise distribution.

It is to be noted that the upper limit $M$ of the $\bar{n}$ modes in the last term of (f) must be greater than $n$ for a solution to be possible. A value of $M \geq n + 1$ is not inconsistent with expression (e), since it has been assumed that $[L^{(\bar{n})} - c] \rightarrow 0$ as $\bar{n}$ increases. Indeed this new procedure provides a simple means of checking whether or not the loading coefficients have reached a constant value. The value of $M$ is taken as high as is practical. If the loading components remain the same with different values of $n \leq M - 1$, then it is safe to say a constant value has been reached.

The kernel element

$$\bar{k}' = -\left[\frac{1}{2}\bar{k}^{(\bar{m},0)} - \sum_{\bar{n}=1}^{M}\bar{k}^{(\bar{m},\bar{n})}\right]$$

can be evaluated simply, no matter what the value of $M$ is. The chordwise integration over $\theta$ is done in one step as

$$\bar{k}' = \frac{-1}{\pi}\int_0^{\pi}\left[\frac{1}{2}\cot\frac{\theta}{2} - \sum_{\bar{n}=1}^{M}\sin\bar{n}\theta\right]\bar{K}^{(\bar{m})}\sin\theta d\theta \quad (20)$$

Since, as was seen earlier, the $\theta$ dependence of $K$ is expressed as $e^{-ix\cos\theta}$,

$$\Lambda^{(M)}(x) = \frac{-1}{\pi}\int_0^{\pi}\left[\frac{1}{2}\cot\frac{\theta}{2} - \sum_{\bar{n}=1}^{M}\sin\bar{n}\theta\right]\sin\theta e^{-ix\cos\theta}d\theta$$
$$= -\frac{1}{2}\Lambda^{(1)}(x) + \sum_{\bar{n}=2}^{M}\Lambda^{(\bar{n})}(x) \quad (21)$$

where

$$\Lambda^{(1)}(x) = J_0(x) - iJ_1(x)$$
$$\Lambda^{(\bar{n})}(x) = [(-i)^{\bar{n}-2}/2][J_{\bar{n}-2}(x) + J_{\bar{n}}(x)] \qquad \bar{n} > 1$$

There will be cancellations of terms in the summation and finally

$$\Lambda^{(M)}(x) = [-(-i)^{M-1}/2][J_{M-1}(x) - iJ_M(x)] \quad (22)$$

The remaining components of the modified kernel function $\bar{K}$ are independent of maximum $\bar{n} = M$; therefore increasing $M$ does not involve additional calculation.

## Application of the Generalized Lift-Operator Technique to Two-Dimensional Nonstationary Problems

It will be demonstrated in this section that use of the generalized lift-operator technique in conjunction with the Birnbaum chordwise modes yields the same results as the known analytical solutions of several two-dimensional nonstationary airfoil problems for incompressible flow: 1) an airfoil advancing at constant speed in a sinusoidal gust $[V_0e^{i\omega(t-x/U)}]$, 2) an airfoil performing heaving oscillation $(-he^{i\omega t})$, and 3) a foil performing rotational oscillation about the midchord point $(\alpha_0 xe^{i\omega t})$.

The two-dimensional counterpart of the integral equation (1) is obtained by distributing pressure dipoles uniformly from $\eta = -\infty$ to $\eta = +\infty$. This leads to

$$W(x)e^{i\omega t} = \int_{-1}^{1}L(\xi)K(x,\xi;t)d\xi \quad (23)$$

where

$$K(x,\xi;t) = \frac{1}{4\pi\rho U}\left\{\lim_{(z-\zeta)\to 0}\int_{\eta=-\infty}^{\infty}\frac{\partial^2}{\partial z^2} \times \int_{\tau=-\infty}^{x-\xi}\frac{e^{ik(\tau-x+\xi)}\,d\tau d\eta}{[\tau^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}}\right\}$$

and all the linear dimensions are nondimensionalized with respect to semichord $c = 1$, and reduced frequency $k$ is defined as

$$k = \omega c/U = \omega/U$$

Reference 7 has shown that the kernel reduces to

$$K(x,\xi;t) = -\frac{1}{2\pi\rho U}\left[-\frac{1}{x-\xi} + ike^{-ik(x-\xi)}\int_{-\infty}^{x-\xi}\frac{e^{ikt}}{t}dt\right] \quad (24)$$

If the Birnbaum modes are taken for the chordwise loading $L(\xi)$ and the lift operator is applied to both sides of Eq. (23), this can be written, for each lift-operator mode $\bar{m}$, in the form

$$\bar{W}^{(\bar{m})} = \sum_{\bar{n}=1}L^{(\bar{n})}\bar{K}^{(m,\bar{n})} \quad (25)$$

where $\bar{K}^{(\bar{m},\bar{n})}$ is the modified kernel after the chordwise integrations, i.e.,

$$\bar{K}^{(\bar{m},\bar{n})} = \frac{1}{\pi^2}\int_0^{\pi}\int_0^{\pi}\Phi(\bar{m})\Theta(\bar{n})K\sin\theta d\theta d\varphi$$

and (26)

$$\bar{W}^{(m)} = \frac{1}{\pi}\int_0^{\pi}\Phi(\bar{m})Wd\varphi$$

(compare preceding sections).

The $\theta$ integration of the kernel is for

$$\bar{n} = 1 \qquad \frac{1}{\pi}\int_0^{\pi}\cot\frac{\theta}{2}\sin\theta K d\theta \equiv \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1-\xi}{1+\xi}\right)^{1/2}Kd\xi$$

$$\bar{n} = 2 \qquad \frac{1}{\pi}\int_0^{\pi}\sin^2\theta K d\theta \equiv \frac{1}{\pi}\int_{-1}^{1}(1-\xi^2)^{1/2}Kd\xi \quad (27)$$

$$\bar{n} = 3 \qquad \frac{1}{\pi}\int_0^{\pi}\sin 2\theta\sin\theta K d\theta \equiv \frac{-2}{\pi}\int_{-1}^{1}\xi(1-\xi^2)^{1/2}Kd\xi$$

$$\bar{n} = 4 \qquad \frac{1}{\pi}\int_0^{\pi}\sin 3\theta\sin\theta K d\theta \equiv \frac{1}{\pi}\int_{-1}^{1}(4\xi^2 - 1)(1-\xi^2)^{1/2}Kd\xi$$

etc. and the $\varphi$ integration is for

$$\bar{m} = 1 \qquad \frac{1}{\pi}\int_0^\pi (1 - \cos\varphi)Kd\varphi \equiv \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{1/2}Kdx$$

$$\bar{m} = 2 \qquad \frac{1}{\pi}\int_0^\pi (1 + 2\cos\varphi)Kd\varphi \equiv \frac{1}{\pi}\int_{-1}^{1}\frac{1-2x}{(1-x^2)^{1/2}}Kdx$$

$$\bar{m} = 3 \qquad \frac{1}{\pi}\int_0^\pi \cos 2\varphi K d\varphi \equiv \frac{1}{\pi}\int_{-1}^{1}\frac{2x^2-1}{(1-x^2)^{1/2}}Kdx$$

etc. The values of $\bar{K}^{(\bar{m},\bar{n})}$ for $\bar{m}$ and $\bar{n}$ from 1 to 4 are obtained in Appendix B of Ref. 8 and are presented in Table 1.

**a. Airfoil in a Sinusoidal Gust**

In this case, the incident velocity distribution is given by

$$W(x)e^{i\omega t} = V_0 e^{i\omega[t - (x/U)]}$$

which, after application of the lift operator, becomes

$$\bar{W}^{(\bar{m})} = V_0 I^{(\bar{m})}(k)$$

where $I^{(\bar{m})}(k)$ is as defined in Appendix A. Therefore, for each order of lift operator, Eq. (25) can be written as

$$V_0 I^{(\bar{m})}(k) = \sum_{\bar{n}=1} L^{(\bar{n})}\bar{K}^{(\bar{m},\bar{n})} \tag{28}$$

If only the first (flat plate) Birnbaum mode and the first-order (Glauert) lift operator are used, Eq. (28) becomes

$$V_0[J_0(k) - iJ_1(k)] = L^{(1)}\bar{K}^{(1,1)}$$

and, with $\bar{K}^{(1,1)}$ taken from Table 1, the loading $L$ is seen to be

$$L = 2\pi\rho\, UV_0\{1/ik[K_0(ik) + K_1(ik)]\} = 2\pi\rho UV_0 S(k)$$

where $S(k)$ is the Sears function.

If maximum $\bar{m}$ and $\bar{n}$ are greater than 1, the equations to be solved are

$$V_0[J_0(k) - iJ_1(k)] = L^{(1)}\bar{K}^{(1,1)} + L^{(2)}\bar{K}^{(1,2)} + \ldots$$

$$V_0[J_0(k) + 2iJ(k)] = L^{(1)}\bar{K}^{(2,1)} + L^{(2)}\bar{K}^{(2,2)} + \ldots$$

$$V_0 I^{(\bar{m})}(k) = L^{(1)}\bar{K}^{(\bar{m},1)} + L^{(2)}\bar{K}^{(\bar{m},2)} + \ldots$$

It can be shown that

$$L^{(1)} = 2\pi\rho UV_0 S(k)$$
$$L^{(2)} = L^{(3)} = \ldots = L^{(\bar{n})} = 0 \tag{29}$$

The total lift is then

$$L = \frac{1}{\pi}\int_0^\pi \left[L^{(1)}\cot\frac{\theta}{2} + \sum_{\bar{n}=2}^{\infty} L^{(\bar{n})}\sin(\bar{n}-1)\theta\right] \times \sin\theta d\theta = L^{(1)} + \frac{1}{2}L^{(2)}$$
$$L = 2\pi\rho UV_0 S(k) \tag{30}$$

Thus the lift obtained through the mathematical model that uses the Birnbaum chordwise modes in conjunction with the generalized lift operator is identical with Sears'[9] lift for an airfoil in a sinusoidal gust. Furthermore, it is interesting to note that only the flat plate mode contributes to the lift.

**Table 1 Values of $\bar{K}(\bar{m}, \bar{n})$ for $\bar{m}$ and $\bar{n}$ from 1 to 4**

| $\bar{m}$ | $\bar{n}$ | $\bar{K}(\bar{m},\bar{n})(-2\pi\rho U)^a$ |
|---|---|---|
| 1 | 1 | $A[J_0(k) - iJ_1(k)]$ |
| 1 | 2 | $B[J_0(k) - iJ_1(k)] + (i/k)$ |
| 1 | 3 | $C[J_0(k) - iJ_1(k)] - (2i/k) + (4/k^2)$ |
| 1 | 4 | $D[J_0(k) - iJ_1(k)] + (3i/k) - (12/k^2) - (24i/k^3)$ |
| 2 | 1 | $A[J_0(k) + 2iJ_1(k)]$ |
| 2 | 2 | $B[J_0(k) + 2iJ_1(k)] + (i/k)$ |
| 2 | 3 | $C[J_0(k) + 2iJ_1(k)] + (4i/k) + (4/k^2)$ |
| 2 | 4 | $D[J_0(k) + 2iJ_1(k)] + (3i/k) + (24/k^2) - (24i/k^3)$ |
| 3 | 1 | $A[-J_2(k)]$ |
| 3 | 2 | $B[-J_2(k)]$ |
| 3 | 3 | $C[-J_2(k)]$ |
| 3 | 4 | $D[-J_2(k)] + (3i/k)$ |
| 4 | 1 | $A[-iJ_3(k)]$ |
| 4 | 2 | $B[-iJ_3(k)]$ |
| 4 | 3 | $C[-iJ_3(k)]$ |
| 4 | 4 | $D[-iJ_3(k)]$ |

[a] $A = -ik[K_0(ik) + K_1(ik)]$, $B = K_1(ik)$, $C = 2[-(2i/k)K_1(ik) + K_0(ik)]$, $D = 3\{[1 - (8/k^2)]K_1(ik) - (4i/k)K_0(ik)\}$.

**b. The Heaving Foil**

For the case of a foil performing heaving oscillations $-he^{i\omega t}$, the downwash distribution is given as

$$W = -i\omega h e^{i\omega t}$$

In this case, Eq. (25) becomes for each $\bar{m}$

$$-ikhUI^{(\bar{m})}(0) = \sum_{\bar{n}=1} L^{(\bar{n})}\bar{K}^{(\bar{m},\bar{n})}$$

where

$$I^{(\bar{m})}(0) = \begin{cases} 1 & \text{for} \quad \bar{m} = 1,2 \\ 0 & \text{for} \quad \bar{m} > 2 \end{cases}$$

When only the first modes are taken ($\bar{m} = \bar{n} = 1$)

$$-ikh\, U = L\bar{K}^{(1,1)}$$

and

$$L = \frac{-2\pi\rho U^2(ikh)}{ik[K_0(ik) + K_1(ik)][J_0(k) - iJ_1(k)]}$$
$$= -2\pi\rho\, U^2 ikh\{S(k)/[J_0(k) - iJ_1(k)]\}$$

Since $S(k) \equiv C(k)[J_0(k) - iJ_1(k)] + iJ_1(k)$ and $C(k) =$ Theodorsen function,

$$L = -2\pi\rho U^2 ikh(C(k) + \{iJ_1(k)/[J_0(k) - iJ_1(k)]\}) \tag{31}$$

This is not equal to Fung's[10] result except for small $k$ when

$$\{iJ_1(k)/[J_0(k) - iJ_1(k)]\} \rightarrow ik/2$$

Taking the first two modes yields the two simultaneous equations

$$-ikhU = L^{(1)}\bar{K}^{(1,1)} + L^{(2)}\bar{K}^{(1,2)}$$
$$-ikhU = L^{(1)}\bar{K}^{(2,1)} + L^{(2)}\bar{K}^{(2,2)}$$

for which the solution is

$$L^{(1)} = -2\pi\rho U^2 hk^2 K_1(ik)/-ik[K_0(ik) + K_1(ik)] = -2\pi\rho U^2 hk[iC(k)]$$
$$L^{(2)} = 2\pi\rho U^2 hk^2 \tag{32}$$

and the total lift is

$$L = L^{(1)} + [L^{(2)}/2] = -2\pi\rho U^2 hk[iC(k) - (k/2)] \tag{33}$$

This is identical with the result of Fung. If maximum ($\bar{m} = \bar{n}) \geq 3$ the results can be shown to be the same as for the

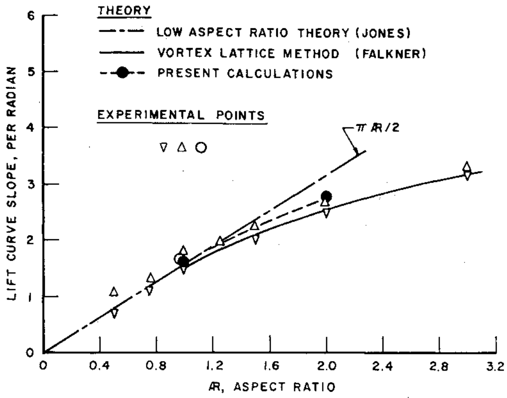



**Fig. 2 Comparison of experimental and theoretical values of the lift-curve slope for untapered wings of 0° sweep (steady state).**

two-mode case with

$$L^{(\bar{n})} = 0 \qquad \text{for} \qquad \bar{n} \geq 3$$

Thus the lift of a heaving foil is attributed to both first and second chordwise modes.

### c. The Pitching Foil

For a foil performing rotational oscillations of small amplitude $+ \alpha_0 x e^{i\omega t}$ about the midchord point, the corresponding downwash distribution is given as

$$W = +\alpha_0 U(ikx + 1)e^{i\omega t}$$

and the left-hand side of Eq. (25) becomes

$$W^{(m)} = \frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m})(\alpha_0 U - \alpha_0 Uik\cos\varphi)d\varphi = \alpha_0 UI^{(\bar{m})}(0) - i\alpha_0 Uk\left[\frac{1}{\pi}\int_0^{\pi}\Phi(\bar{m})\cos\varphi d\varphi\right]$$

where

$$\frac{1}{\pi}\int_0^{\pi}\Phi(\bar{m})\cos\varphi d\varphi = \begin{cases} -\frac{1}{2} & \text{for } \bar{m} = 1 \\ +1 & \text{for } \bar{m} = 2 \\ 0 & \text{for } \bar{m} > 2 \end{cases}$$

Then Eq. (25) for the pitching foil becomes

$$\alpha_0 U\left(1 + \frac{ik}{2}\right) = \sum_{\bar{n}=1} L^{(\bar{n})}\bar{K}^{(1,\bar{n})} \qquad \bar{m} = 1$$

$$\alpha_0 U(1 - ik) = \sum_{\bar{n}=1} L^{(\bar{n})}\bar{K}^{(2,\bar{n})} \qquad \bar{m} = 2 \qquad (34)$$

$$0 = \sum_{\bar{n}=1} L^{(\bar{n})}\bar{K}^{(\bar{m},\bar{n})} \qquad \bar{m} > 2$$

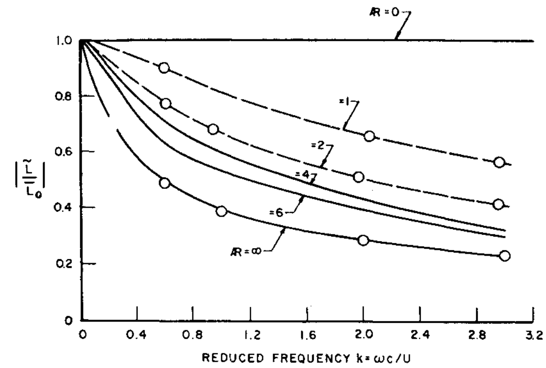
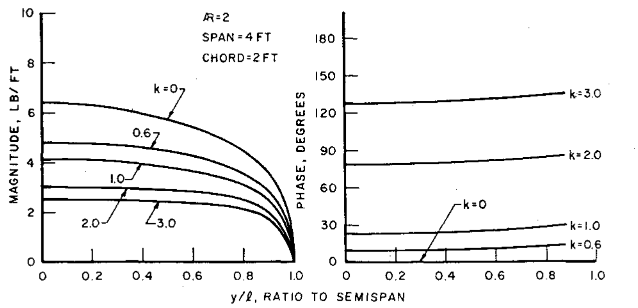



**Fig. 3 Ratio of unsteady to quasi-steady lift response of rectangular foils to sinusoidal gust of unit amplitude (solid curves taken from Ref. 13, circles are present calculations).**




**a) Magnitudes** **b) Phases**

**Fig. 4 Spanwise distribution of lift response of rectangular foil of aspect ratio 2 to sinusoidal gust of unit amplitude.**

When $\bar{m} = \bar{n} = 1$

$$L = \frac{\alpha_0 U[1 + (ik/2)](2\pi\rho U)}{ik[K_0(ik) + K_1(ik)][J_0(k) - iJ_1(k)]} = 2\pi\rho U^2\alpha_0\left(1 + \frac{ik}{2}\right)\left[C(k) + \frac{iJ_1(k)}{J_0(k) - iJ_1(k)}\right] \qquad (35)$$

which approaches Fung's result only for small $k$.

When two modes are taken, it is easily shown that the total lift again approaches Fung's lift only for small $k$. When three or more modes are taken, it can be shown that

$$L^{(1)} = 2\pi\alpha_0\rho U^2\{[1 + (ik/2)]C(k) - (ik/2)\}$$

$$L^{(2)} = 2\pi\alpha_0\rho U^2(2ik)$$

$$L^{(3)} = 2\pi\alpha_0\rho U^2(k^2/4)$$

$$L^{(\bar{n})} = 0 \qquad \text{for} \qquad \bar{n} > 3$$

and the total lift is

$$L = 2\pi\alpha_0\rho U^2\{C(k)[1 + (ik/2)] + (ik/2)\} \qquad (36)$$

which is identical with Fung's result for the case of a pitching foil. In this case, three chordwise modes contribute to the total lift.

## Numerical Results for Finite-Aspect-Ratio Foils

A computer program adapted to the CDC-3600 digital computer has been devised for the numerical solution of the surface integral equation. The program, which follows the development described in preceding sections, has been tested by application to the case of the lift response of two rectangular foils, of aspect ratio 1 and 2, to a sinusoidal gust of unit amplitude. The calculated results have been compared with available experimental data and the results of other theoretical methods.

Figure 2 is a comparison, such as that shown in Fig. A, 7t of Ref. 11, between experimental values of the lift-curve slope (steady state) and the theoretical values obtained by low-aspect-ratio theory,[11] by Falkner's vortex-lattice method[12] and by the present procedure. The experimental data for rectangular foils were taken from Ref. 11. As is seen, the results of the present lifting surface approach fall close to the upper limits of the experimental range, whereas the results of the vortex-lattice method follow the lower limits of the experimental data.

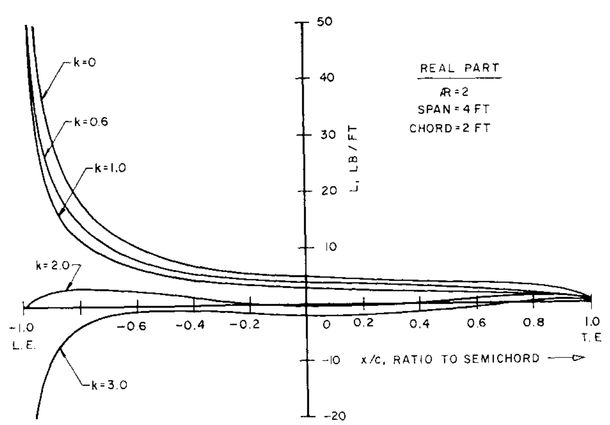
Figure 3 shows the ratio of gust-generated amplitudes of unsteady to quasi-steady lift as functions of reduced frequency and aspect ratio. The solid curves are taken from the "unsteady-lift functions" evaluated by Drischler[13] for rigid foils of aspect ratio 4, 6, and ∞ in a sinusoidal gust in the case of incompressible flow. The circles on Fig. 3 are the present calculations for foils of aspect ratio 1, 2, and ∞.

The figures following present detailed results of the present calculations. Figures 4a and 4b show the spanwise distribution of the lift response for a foil of aspect ratio 2 (4-ft span and 2-ft chord). Figure 4a depicts the magnitudes at various frequencies and 4b the corresponding phases. Figures 5a and 5b show real and imaginary parts, respectively, of the chordwise distribution at one spanwise location, for various reduced frequencies.

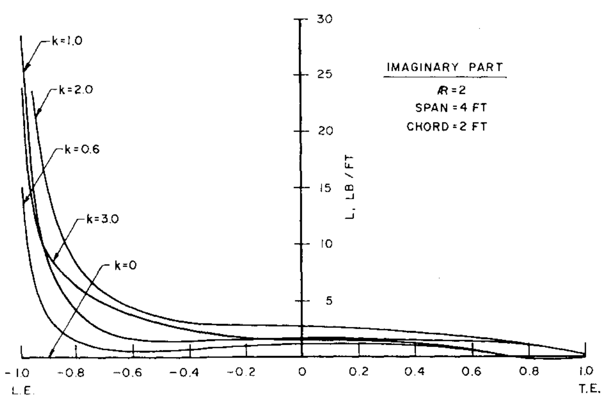
Figures 6a–6d present the results for a foil of aspect ratio 1 (4-ft span and 4-ft chord). The plots in Figs. 6a–6d are spanwise distributions at various chordwise positions, unlike the integrated spanwise distributions shown in Figs. 4a and 4b. Each chart (Figs. 6a–6d) is for a different frequency $k$ and exhibits the magnitudes and corresponding phases.

## Conclusion

A new method of inverting the integral equation for lifting surfaces has been developed and applied to cases of foils of finite aspect-ratio advancing at constant speed in a sinusoidal gust. The method employs a generalized lift-operator technique which is found to be a more accurate, versatile, and rapid procedure than the presently used "mode-collocation" approach. This new method has also been applied to problems of two-dimensional unsteady airfoil theory, such as a foil advancing in a sinusoidal gust or undergoing prescribed motions, demonstrating its capability of obtaining the known explicit solutions. A computer program adapted to the CDC-3600 digital computer has been devised, with capability of determining the steady and unsteady pressure distributions and corresponding hydrodynamic forces and moments on lifting surfaces of arbitrary planform undergoing arbitrary motion.



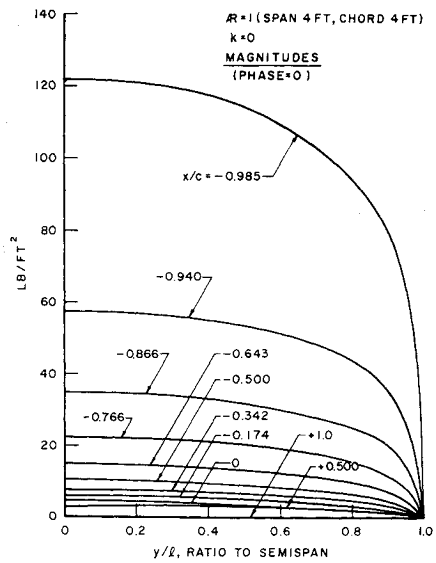

a) Real part



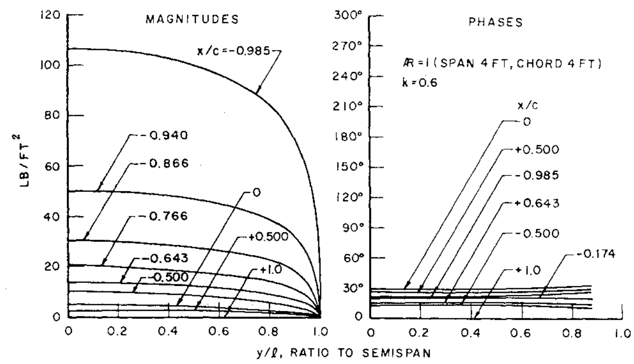

b) Imaginary part

**Fig. 5 Chordwise distribution of lift response of rectangular foil of aspect ratio 2 to sinusoidal gust of unit amplitude at $y/l = \pm 0.125$.**

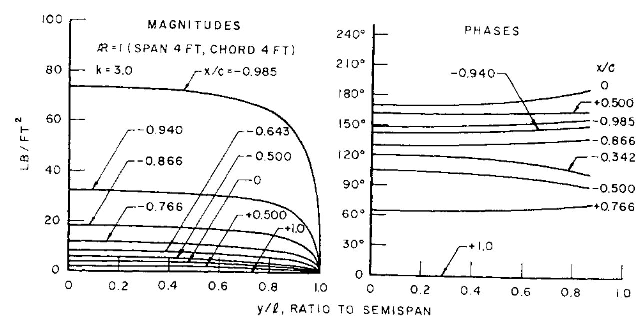
**Fig. 6 Spanwise distribution of lift response of rectangular foil of aspect ratio 1 to sinusoidal gust of unit amplitude at various chordwise locations $x/c$.**



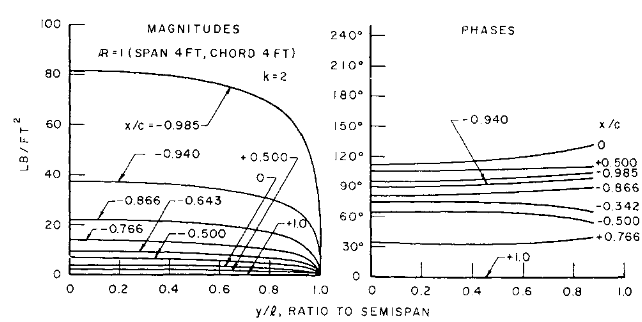

a) Reduced frequency $k = 0$




b) $k = 0.6$




c) $k = 2.0$




d) $k = 3.0$

## Appendix A: Values of $I^{(\bar{m})}(\nu)$ and $\Lambda^{(\bar{n})}(\nu)$ for Various $\bar{m}$, $\bar{n}$

$$\left.\begin{aligned}
I^{(\bar{m})}(\nu) &= \frac{1}{\pi}\int_0^{\pi} \Phi(\bar{m})^{i\nu\cos\varphi}\, d\varphi \\
I^{(1)}(\nu) &= \frac{1}{\pi}\int_0^{\pi} (1-\cos\varphi)e^{i\nu\cos\varphi}\, d\varphi = J_0(\nu) - iJ_1(\nu) \\
I^{(2)}(\nu) &= \frac{1}{\pi}\int_0^{\pi} (1+2\cos\varphi)e^{i\nu\cos\varphi}\, d\varphi = J_0(\nu) + 2iJ_1(\nu) \\
I^{(\bar{m})}(\nu) &= \frac{1}{\pi}\int_0^{\pi} \cos(\bar{m}-1)\varphi\, e^{i\nu\cos\varphi}\, d\varphi \quad \text{for } \bar{m} > 2 \\
&= i^{(\bar{m}-1)}J_{\bar{m}-1}(\nu)
\end{aligned}\right\} \quad \text{(A1)}$$

$$\left.\begin{aligned}
\Lambda^{(\bar{n})}(\nu) &= \frac{1}{\pi}\int_0^{\pi} \Theta(\bar{n})e^{-i\nu\cos\theta}\sin\theta d\theta \\
\Lambda^{(1)}(\nu) &= \frac{1}{\pi}\int_0^{\pi} \cot\frac{\theta}{2}\sin\theta\, e^{-i\nu\cos\theta}\, d\theta = J_0(\nu) - iJ_1(\nu) \\
\Lambda^{(\bar{n})}(\nu) &= \frac{1}{\pi}\int_0^{\pi} \sin(\bar{n}-1)\theta\,\sin\theta\, e^{-i\nu\cos\theta}\, d\theta = \\
&\frac{(-i)^{(\bar{n}-2)}}{2}\,[J_{n-2}(\nu) + J_{\bar{n}}(\nu)] \qquad \bar{n} > 1
\end{aligned}\right\} \quad \text{(A2)}$$

The values for $(-\nu)$ are the complex conjugates of the foregoing expressions for $(+\nu)$.

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